An Arctangent Series

11932 [2016, 831]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let r be an integer. Prove

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r.$$

Solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, Logroño, Spain. It suffices to prove the result for $r > 0$, because the functions sinh and arctan are odd, and the case $r = 0$ is easy. Using the identity

$$\arctan\left(\frac{x-y}{1+xy}\right) = \arctan x - \arctan y,$$

we obtain

$$\arctan\left(\frac{\sinh\ r}{\cosh\ n}\right) = \arctan\ \frac{e^{-(n-r)} - e^{-(n+r)}}{1+e^{-2n}} = \arctan\ e^{-(n-r)} - \arctan\ e^{-(n+r)}.$$

Let $b_n = \arctan(e^{-(n+r)})$, and let S be the sum to be evaluated. We have

$$\begin{aligned} S - \arctan(\sinh\ r) &= 2\sum_{n=1}^{\infty} \arctan\ \left(\frac{\sinh\ r}{\cosh\ n}\right) = 2\lim_{N\to\infty} \sum_{n=1}^{N} (b_{n-2r} - b_n) \\ &= 2\sum_{n=1}^{2r} b_{n-2r} - 2\lim_{N\to\infty} \sum_{n=N+1}^{N+2r} b_{n-2r} = 2\sum_{n=1}^{2r} b_{n-2r}. \end{aligned}$$

Using the identity $\arctan z + \arctan\ z^{-1} = \pi/2$ for $z > 0$, we deduce

$$\begin{aligned}\sum_{n=1}^{2r} b_{n-2r} &= \sum_{n=1}^{r} \arctan\left(e^{-(n-r)}\right) + \sum_{n=r+1}^{2r} \arctan\left(e^{-(n-r)}\right)\\ &= \sum_{m=1}^{r-1}\left(\arctan(e^{m}) + \arctan(e^{-m})\right) + \arctan(1) + \arctan(e^{-r})\\ &= \frac{\pi}{2}(r-1) + \frac{\pi}{4} + \arctan(e^{-r}).\end{aligned}$$

Substituting this sum into S, we obtain

$$\begin{aligned}S &= \arctan(e^{r}) - \arctan(e^{-r}) + 2\left(\frac{\pi}{2}(r-1) + \frac{\pi}{4} + \arctan(e^{-r})\right)\\ &= \frac{\pi}{2} + \pi(r-1) + \frac{\pi}{2} = \pi r.\end{aligned}$$

Also solved by A. Berhane (Algeria), R. Boukaharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), L. Glasser, J. Grzesik, A. Harnist (France), E. Ionascu, W. Johnson, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), K. Kolczyńska-Przybycień (Poland), K. Koo (China), O. Kouba (Syria), P. Magli (Italy), R. Molinari, R. Nandan, M. Omarjee (France), Á. Plaza (Spain), M. A. Prasad (India), N. Singer, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), C. Vălean (Romania), G. Vidiani (France), M. Vowe (Switzerland), J. Zacharias, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.