A HHO method for incompressible flows of non-Newtonian fluids with power-like convective behaviour

with Daniel Castanon Quiroz and Daniele A. Di Pietro

André Harnist

andreharnist.fr

Institut Montpelliérain Alexander Grothendieck – University of Montpellier

18th EFEF – INRIA, Paris – Sept 10-11, 2021

Generalized Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, open connected polyhedral with Lipschitz boundary $\partial\Omega$.

Generalized Navier–Stokes problem: Find $(\boldsymbol{u}, p)$ such that

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} &\quad \text{in } \Omega,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 &\quad \text{in } \Omega,\\
\boldsymbol{u} = \boldsymbol{0} &\quad \text{on } \partial\Omega,\\
\int_\Omega p = 0. &
\end{aligned}
$$

- Force $\boldsymbol{f} : \Omega \to \mathbb{R}^d$
- Velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^d$
- Pressure $p : \Omega \to \mathbb{R}$
- Viscosity law $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_{\mathrm{s}}^{d\times d} \to \mathbb{R}_{\mathrm{s}}^{d\times d}$

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Generalized Navier–Stokes problem: Find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} -\boldsymbol{\nabla}\cdot \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\chi}(\cdot, \boldsymbol{u}) + \boldsymbol{\nabla} p &= \boldsymbol{f} \quad &&\text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 &&\text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} &&\text{on } \partial\Omega, \\ \int_\Omega p &= 0. \end{aligned}$$

- Force $\boldsymbol{f} : \Omega \to \mathbb{R}^d$
- Velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^d$
- Pressure $p : \Omega \to \mathbb{R}$
- Viscosity law $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$
- Convection law $\boldsymbol{\chi} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *Idea!*

Assumptions on the viscosity law $\boldsymbol{\sigma}$

For all $m \in (1, \infty)$, we define its conjugate exponent $m' = \frac{m}{m-1}$.

Assumption. *(Viscosity law)*

There exists $r \in (1, \infty)$, $\delta \in L^r(\Omega, [0, +\infty))$ and $\sigma_{\rm hc}, \sigma_{\rm hm} \in (0, +\infty)$ s.t.

$$\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) \text{ is measurable and } \boldsymbol{\sigma}(\boldsymbol{x}, \mathbf{0}) \in L^{r'}(\Omega, \mathbb{R}_{\rm s}^{d\times d}),$$

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})|_{d\times d} \leqslant \sigma_{\rm hc} \left(\delta(\boldsymbol{x})^r + |\boldsymbol{\tau}|_{d\times d}^r + |\boldsymbol{\eta}|_{d\times d}^r\right)^{\frac{r-2}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d},$$

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \sigma_{\rm hm} \left(\delta(\boldsymbol{x})^r + |\boldsymbol{\tau}|_{d\times d}^r + |\boldsymbol{\eta}|_{d\times d}^r\right)^{\frac{r-2}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d}^2,$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}_{\rm s}^{d\times d}$ and a.e. $\boldsymbol{x} \in \Omega$.

Assumptions on the convection law χ

For all $m \in (1,\infty)$, we define its singular exponent $\tilde{m} = \min(m,2)$.

Assumption. *(Convection law)*

There exists $s \in (1,\infty)$ and $\chi_{\mathrm{hc}} \in (0,+\infty)$ s.t.

$$\boldsymbol{\chi}(\cdot,\boldsymbol{w}) \text{ is measurable and } \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0},$$

$$|\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{w}) - \boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{v})| \leqslant \chi_{\mathrm{hc}} \left(|\boldsymbol{w}|^s + |\boldsymbol{v}|^s\right)^{\frac{s-\tilde{s}}{s}} |\boldsymbol{w}-\boldsymbol{v}|^{\tilde{s}-1},$$

$$(\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{\chi}(\cdot,\boldsymbol{w}) = (\boldsymbol{\chi}(\cdot,\boldsymbol{w})\cdot\boldsymbol{\nabla})\boldsymbol{w} + (s-2)\frac{(\boldsymbol{\chi}(\cdot,\boldsymbol{w})\cdot\boldsymbol{\nabla})\boldsymbol{w}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}\boldsymbol{w},$$

$$\boldsymbol{w}\otimes\boldsymbol{\chi}(\cdot,\boldsymbol{w}) = \boldsymbol{\chi}(\cdot,\boldsymbol{w})\otimes\boldsymbol{w}.$$

for all $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Omega$.

Weak formulation

Assume $\boldsymbol{f} \in L^{r'}(\Omega)^d$.
Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U} \coloneqq W_0^{1,r}(\Omega)^d, \\ -b(\boldsymbol{u},q) &= 0 && \forall q \in P \coloneqq L_0^{r'}(\Omega), \end{aligned}$$

where,

$$\begin{aligned} a(\boldsymbol{w},\boldsymbol{v}) &\coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{w}) : \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}, \\ b(\boldsymbol{v},q) &\coloneqq -\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{v})q, \\ c(\boldsymbol{w},\boldsymbol{v}) &\coloneqq \frac{1}{s}\int_\Omega (\boldsymbol{\chi}(\cdot,\boldsymbol{w})\cdot\boldsymbol{\nabla})\boldsymbol{w}\cdot\boldsymbol{v} - \frac{1}{s'}\int_\Omega (\boldsymbol{\chi}(\cdot,\boldsymbol{w})\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{w} \\ &\qquad + \frac{s-2}{s}\int_\Omega \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|^2}(\boldsymbol{\chi}(\cdot,\boldsymbol{w})\cdot\boldsymbol{\nabla})\boldsymbol{w}\cdot\boldsymbol{w}. \end{aligned}$$

Weak formulation

Assume $\boldsymbol{f} \in L^{r'}(\Omega)^d$.
Weak formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U} := W_0^{1,r}(\Omega)^d, \\ -b(\boldsymbol{u}, q) &= 0 && \forall q \in P := L_0^{r'}(\Omega), \end{aligned}$$

where,

$$\begin{aligned} a(\boldsymbol{w}, \boldsymbol{v}) &:= \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}) : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}, \\ b(\boldsymbol{v}, q) &:= -\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{v}) q, \\ c(\boldsymbol{w}, \boldsymbol{v}) &:= \frac{1}{s} \int_\Omega (\boldsymbol{\chi}(\cdot, \boldsymbol{w}) \cdot \boldsymbol{\nabla}) \boldsymbol{w} \cdot \boldsymbol{v} - \frac{1}{s'} \int_\Omega (\boldsymbol{\chi}(\cdot, \boldsymbol{w}) \cdot \boldsymbol{\nabla}) \boldsymbol{v} \cdot \boldsymbol{w} \\ &\quad + \frac{s-2}{s} \int_\Omega \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{|\boldsymbol{w}|^2} (\boldsymbol{\chi}(\cdot, \boldsymbol{w}) \cdot \boldsymbol{\nabla}) \boldsymbol{w} \cdot \boldsymbol{w}. \end{aligned}$$

Non-dissipativity: $c(\boldsymbol{w}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \boldsymbol{U}$.

Discrete weak formulation

Discrete weak formulation: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that

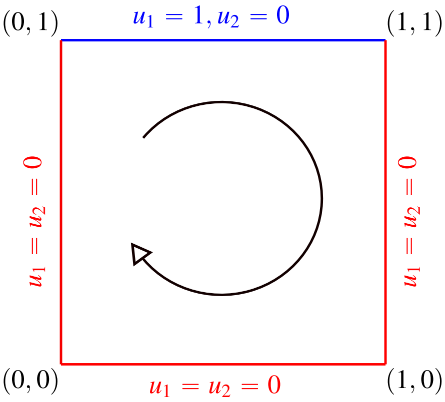
$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{c}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in P_h^k, \end{aligned}$$

where,

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) &:= \int_\Omega \boldsymbol{\sigma}(\cdot, \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{w}}_h) : \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h), \\ \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) &:= -\int_\Omega \mathrm{D}_h^k \underline{\boldsymbol{v}}_h \, q_h, \\ \mathrm{c}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) &:= \frac{1}{s} \int_\Omega (\boldsymbol{\chi}(\cdot, \boldsymbol{w}_h) \cdot \mathbf{G}_h^k) \underline{\boldsymbol{w}}_h \cdot \boldsymbol{v}_h - \frac{1}{s'} \int_\Omega (\boldsymbol{\chi}(\cdot, \boldsymbol{w}_h) \cdot \mathbf{G}_h^k) \underline{\boldsymbol{v}}_h \cdot \boldsymbol{w}_h \\ &\quad + \frac{s-2}{s} \int_\Omega \frac{\boldsymbol{v}_h \cdot \boldsymbol{w}_h}{|\boldsymbol{w}_h|^2} (\boldsymbol{\chi}(\cdot, \boldsymbol{w}_h) \cdot \mathbf{G}_h^k) \underline{\boldsymbol{w}}_h \cdot \boldsymbol{w}_h. \end{aligned}$$

Non-dissipativity: $\mathrm{c}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{w}}_h) = 0$ for all $\underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{U}}_h^k$.

Lid-driven cavity problem



With a moderate Reynolds number $\mathrm{Re} = 1000$, we set for all $\boldsymbol{\tau} \in \mathbb{R}_{\mathrm{s}}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = \frac{2}{\mathrm{Re}}(1 + |\boldsymbol{\tau}|_{d\times d}^{r})^{\frac{r-2}{r}}\boldsymbol{\tau} \quad \text{and} \quad \boldsymbol{\chi}(\boldsymbol{w}) = |\boldsymbol{w}|^{s-2}\boldsymbol{w}.$$

Lid-driven cavity problem

We set $s = 2$ and we vary r.

$r = \frac{3}{2}$

$r = 2$

$r = 3$

Lid-driven cavity problem

We set $s = 2$ and we vary r.

$r = \frac{3}{2}$

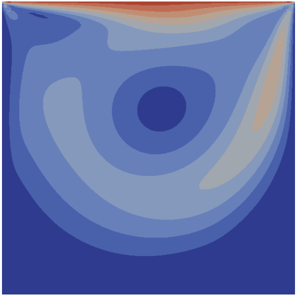
$r = 2$

$r = 3$

Viscous effects increase with r!

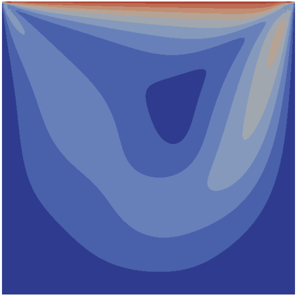
Lid-driven cavity problem

We set $r = \frac{5}{2}$ and we vary s.



$s = \frac{3}{2}$

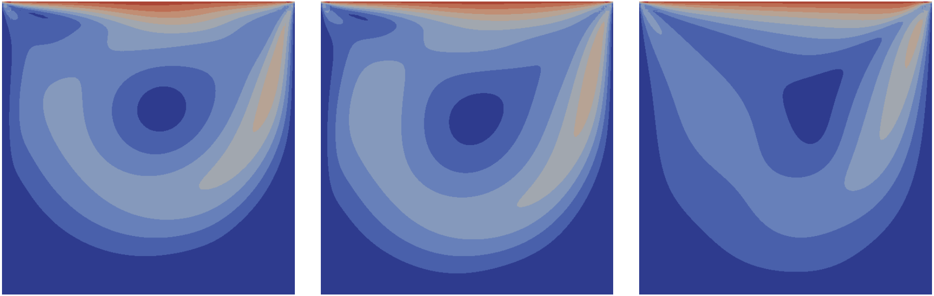
$s = 2$



$s = \frac{9}{2}$

Lid-driven cavity problem

We set $r = \frac{5}{2}$ and we vary s.



$s = \frac{3}{2}$ $\qquad$ $s = 2$ $\qquad$ $s = \frac{9}{2}$

Turbulent effects increase with s!

Main results

For all $m \in [1, \infty]$, we define its Sobolev exponent $m^* := \begin{cases} \frac{dm}{d-m} & \text{if } m < d \\ \infty & \text{if } m \geqslant d \end{cases}$.

Main results. *(Well-posedness, convergence and error estimates)*

- Existence: There exist solutions to the weak and discrete weak formulations for all $r, s \in (1, \infty)$.
- Uniqueness: Assuming $2 \leqslant s \leqslant \frac{\tilde{r}^*}{\tilde{r}'}$ and a data smallness condition yields the uniqueness of the solutions.
- Convergence: Assuming $s < \frac{r^*}{r'}$ yields convergence results to minimal regularity solutions.
- Error estimate: Assuming $r \leqslant 2 \leqslant s \leqslant \frac{r^*}{r'}$, uniqueness of the solutions, and additional regularity yields error estimates.
  Convergence rates in $[(k+1)(r-1), k+1]$ for the velocity, and $[(k+1)(r-1)^2, (k+1)(r-1)]$ for the pressure, according to δ.

Thank you for your attention!

[1] M. Botti, D. Castanon Quiroz, D. A. Di Pietro, and A. Harnist. “A Hybrid High-Order method for creeping flows of non-Newtonian fluids”. In: *ESAIM: Math. Model Numer. Anal.* (2021). Accepted for publication. URL: https://hal.archives-ouvertes.fr/hal-02519233.

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[5] Lei L. and Jian-Guo L. “p-Euler equations and p-Navier–Stokes equations”. In: *Journal of Differential Equations* 264.7 (2018), pp. 4707–4748. ISSN: 0022-0396. DOI: 10.1016/j.jde.2017.12.023.