A Hybrid High-Order method for creeping flows of non-Newtonian fluids

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with

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Outline

1. The Hybrid High-Order method

2. Newtonian and non-Newtonian fluids

3. The Stokes equations

4. Discretization with the HHO method

Hybrid High-Order (HHO)

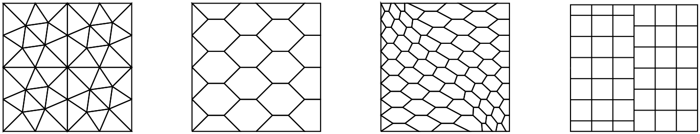
Hybrid: two kinds of unknowns located on the mesh and its skeleton.

High-Order: the unknowns live in broken polynomial spaces of degree $k \in \mathbb{N}$.

General features

This approach possesses several attractive features:

- Arbitrary approximation order ($k \geqslant 0$).
- Formulation valid for arbitrary space dimension.
- Seamless treatment of nonconforming mesh refinement.



- Moderate computational costs thanks to static condensation.
- Inf-sup stable discretizations.

References

Book.

D. A. Di Pietro and J. Droniou.
The Hybrid High-Order method for polytopal meshes.
Design, Analysis, and Applications
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Subject matter

Article.

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Outline

Newtonian and non-Newtonian fluids

We can distinguish fluids according to their viscosity:

- Newtonian: viscosity is constant with increased stress (e.g. air, water).
- non-Newtonian:
  - Pseudoplastic (shear thinning): viscosity decreases with increased stress (e.g. blood, honey).
  - Dilatant (shear thickening): viscosity increases with increased stress (e.g. oobleck, quicksand).

Newtonian fluid

We characterize the movement of a fluid with a strain-stress function

$$\boldsymbol{\sigma} : \Omega \times \mathbb{R}_{\mathrm{s}}^{d\times d} \to \mathbb{R}_{\mathrm{s}}^{d\times d}$$

where $\mathbb{R}_{\mathrm{s}}^{d\times d} := \{\boldsymbol{\tau} \in \mathbb{R}^{d\times d} : \boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}\}$, $d \in \{2, 3\}$.

- A Newtonian fluid is one for which the law $\boldsymbol{\sigma}$ is linear.
- For the non-Newtonian fluids, several laws model them:
  - Power-law
  - Carreau–Yasuda
  - Yeleswarapu
  - Quemada
  - Cross
  - ...

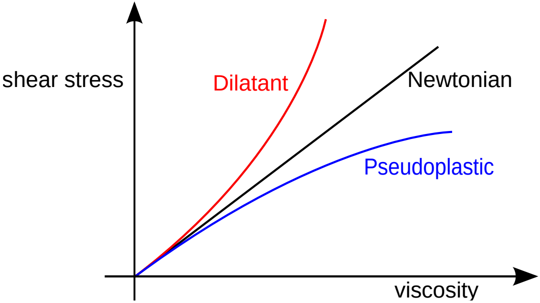
Power-law

A power-law fluid is one for which $\boldsymbol{\sigma}$ is such that,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = \mu|\boldsymbol{\tau}|_{d\times d}^{r-2}\boldsymbol{\tau} \qquad \forall \boldsymbol{\tau} \in \mathbb{R}_{\mathrm{s}}^{d\times d},$$

where $\mu > 0$ is the flow consistency index and $r > 1$ is the flow behavior index.

- If $r < 2$, the fluid is pseudoplastic (shear thinning).
- If $r = 2$, the fluid is Newtonian.
- If $r > 2$, the fluid is dilatant (shear thickening).



Outline

The steady generalized Stokes problem

Let $\Omega \subset \mathbb{R}^d$ denote a bounded, connected polyhedral open set with Lipschitz boundary $\partial\Omega$.

The steady generalized Stokes problem reads: Find $\boldsymbol{u}$ and p such that

$$\begin{aligned} -\boldsymbol{\nabla}\cdot \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \boldsymbol{\nabla} p &= \boldsymbol{f} \quad && \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \\ \int_\Omega p &= 0, \end{aligned}$$

where,

- $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is the force applied on the fluid,
- $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity of the fluid and $\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} \coloneqq \dfrac{\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^{\mathsf{T}}\boldsymbol{u}}{2}$,
- $p : \Omega \to \mathbb{R}$ is the pressure of the fluid,
- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ is the strain-stress law of the fluid.

Assumptions on $\boldsymbol{\sigma}$ - power-framed function

Assumption. *(power-framed)*

There exists $r \in (1, +\infty)$ such that:

- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ is measurable,
- $\boldsymbol{\sigma}(\cdot, \mathbf{0}) \in L^{r'}(\Omega, \mathbb{R}^{d\times d}_{\mathrm{s}})$ a.e. in Ω where $r' \coloneqq \frac{r}{r-1}$,
- $\boldsymbol{\sigma}$ is r-power-framed: defining the singular exponent of r by

$$\tilde{r} \coloneqq \min(r, 2),$$

there is $\sigma_{\mathrm{de}}, \sigma_{\mathrm{hc}}, \sigma_{\mathrm{sm}} \in \mathbb{R}^{+} \times \mathbb{R}^{+}_{*} \times \mathbb{R}^{+}_{*}$ s.t.

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})|_{d\times d} \leqslant \sigma_{\mathrm{hc}} \left(\sigma_{\mathrm{de}}^{r} + |\boldsymbol{\tau}|^{r}_{d\times d} + |\boldsymbol{\eta}|^{r}_{d\times d}\right)^{\frac{r-\tilde{r}}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{\tilde{r}-1}_{d\times d},$$

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \sigma_{\mathrm{sm}} \left(\sigma_{\mathrm{de}}^{r} + |\boldsymbol{\tau}|^{r}_{d\times d} + |\boldsymbol{\eta}|^{r}_{d\times d}\right)^{\frac{\tilde{r}-2}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{r+2-\tilde{r}}_{d\times d},$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{s}}$ and a.e. $\boldsymbol{x} \in \Omega$.

Weak formulation

Assuming $\boldsymbol{f} \in L^{r'}(\Omega, \mathbb{R}^d)$, we define

- $\boldsymbol{U} \coloneqq W_0^{1,r}(\Omega, \mathbb{R}^d) = \left\{\boldsymbol{v} \in W^{1,r}(\Omega, \mathbb{R}^d) \ : \ \boldsymbol{v}_{|\partial\Omega} = \boldsymbol{0}\right\}$,
- $P \coloneqq L_0^{r'}(\Omega, \mathbb{R}) = \left\{q \in L^{r'}(\Omega, \mathbb{R}) \ : \ \int_\Omega q = 0\right\}$.

The weak formulation of the Stokes problem reads:
Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u}, q) &= 0 && \forall q \in P, \end{aligned}$$

where, for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{U}$ and all $q \in P$,

$$a(\boldsymbol{w}, \boldsymbol{v}) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}) : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v},$$

$$b(\boldsymbol{v}, q) \coloneqq -\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{v}) q.$$

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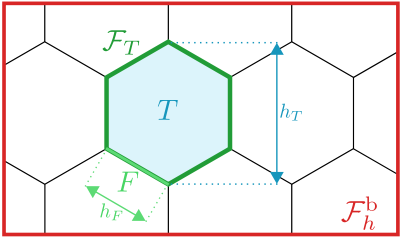
Mesh and notations

Let $h \in (0, +\infty)$. We define a mesh of Ω as a couple $(\mathcal{T}_h, \mathcal{F}_h)$ such that

- $\mathcal{T}_h$ is a finite collection of polyhedral elements T with diameter h_T,
- $\mathcal{F}_h$ is a finite collection of planar faces F with diameter h_F,
- $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$ and $\max_{T\in\mathcal{T}_h} h_T = h$,
- $(\mathcal{T}_h, \mathcal{F}_h)$ satisfies some geometrical requirements...

We also define the following subsets of $\mathcal{F}_h$:

- $\mathcal{F}_h^{\mathrm{b}} \coloneqq \{F \in \mathcal{F}_h \ : \ F \subset \partial\Omega\}$,
- $\mathcal{F}_T \coloneqq \{F \in \mathcal{F}_h \ : \ F \subset \partial T\}$ for all $T \in \mathcal{T}_h$.



Discrete spaces and notations

Let $k \geqslant 1$ be the polynomial degree of the HHO method.

- For all $T \in \mathcal{T}_h$, we define the discrete local space:

$$\underline{\boldsymbol{U}}_T^k \coloneqq \mathbb{P}^k(T, \mathbb{R}^d) \times \left(\prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F, \mathbb{R}^d) \right).$$

  We use the discrete notation $\underline{\boldsymbol{v}}_T \coloneqq (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \in \underline{\boldsymbol{U}}_T^k$.

- We define the discrete global space:

$$\underline{\boldsymbol{U}}_h^k \coloneqq \prod_{T \in \mathcal{T}_h} \underline{\boldsymbol{U}}_T^k.$$

  We use the discrete notation $\underline{\boldsymbol{v}}_h \coloneqq (\underline{\boldsymbol{v}}_T)_{T \in \mathcal{T}_h} \in \underline{\boldsymbol{U}}_h^k$.

- We define the interpolation operator $\underline{\mathbf{I}}_h^k : W^{1,1}(\Omega, \mathbb{R}^d) \to \underline{\boldsymbol{U}}_h^k$ s.t.

$$\underline{\mathbf{I}}_h^k \boldsymbol{v} \coloneqq (\boldsymbol{\pi}_T^k \boldsymbol{v}_{|_T}, (\boldsymbol{\pi}_F^k \boldsymbol{v}_{|_F})_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h}.$$

Spaces and norms of discrete unknowns

- The discrete space containing velocity unknowns is defined by

$$\underline{\boldsymbol{U}}_{h,0}^{k} \coloneqq \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}.$$

We endow $\underline{\boldsymbol{U}}_{h,0}^{k}$ with the semi-norm $\|\cdot\|_{\varepsilon,r,h}$ defined by

$$\|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}^r \coloneqq \sum_{T\in\mathcal{T}_h} \left(\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}_T\|_{L^r(T,\mathbb{R}^{d\times d})}^r + \sum_{F\in\mathcal{F}_T} h_F^{1-r} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^r(F,\mathbb{R}^d)}^r \right).$$

- The discrete space containing pressure unknowns is defined by

$$P_h^k \coloneqq \left\{ q_h \in L_0^{r'}(\Omega,\mathbb{R}) \ : \ (q_h)_{|T} \in \mathbb{P}^k(T,\mathbb{R}) \quad \forall T \in \mathcal{T}_h \right\}.$$

We endow P_h^k with the norm $\|\cdot\|_{L^{r'}(\Omega,\mathbb{R})}$.

Korn and discrete Korn inequalities

The regularity of Ω yields the following Korn inequality:

$$\|\boldsymbol{v}\|_{W^{1,r}(\Omega,\mathbb{R}^d)} \lesssim \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}\|_{L^r(\Omega,\mathbb{R}^{d\times d})} \qquad \forall \boldsymbol{v} \in W_0^{1,r}(\Omega,\mathbb{R}^d).$$

Theorem. *(discrete Korn inequality)*

It holds, with hidden constant depending only on Ω, d, k, ρ and r,

$$\|\boldsymbol{v}_h\|_{W^{1,r}(\mathcal{T}_h,\mathbb{R}^d)} \lesssim \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h} \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k,$$

As a consequence, $\|\cdot\|_{\varepsilon,r,h}$ is a norm on $\underline{\boldsymbol{U}}_{h,0}^k$.

Hilbertian case $r = 2 \rightsquigarrow$ see [Botti, Di Pietro, Guglielmana; 2019]

Discrete operators

For all $T \in \mathcal{T}_h$, we define:

- the discrete local symmetric gradient $\mathbf{G}_{\mathrm{s},T}^k : \underline{\boldsymbol{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d})$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\int_T \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = \int_T \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T : \boldsymbol{\tau} + \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{v}_F - \boldsymbol{v}_T) \cdot (\boldsymbol{\tau} \boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d}).$$

- the discrete local divergence $\mathrm{D}_T^k : \underline{\boldsymbol{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R})$ as the trace of the discrete gradient operator: $\mathrm{D}_T^k = \mathrm{tr}(\mathbf{G}_{\mathrm{s},T}^k)$.

The global versions of these operators are defined by: for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$(\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h)_{|T} := \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathcal{T}_h,$$

$$(\mathrm{D}_h^k \underline{\boldsymbol{v}}_h)_{|T} := \mathrm{D}_T^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathcal{T}_h.$$

Discrete weak formulation

The discrete weak formulation reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ s.t.

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in P_h^k, \end{aligned}$$

where, for all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$ and $q_h \in P_h^k$,

- $\mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{w}}_h) : \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)$,
- $\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) \coloneqq -\int_\Omega \mathrm{D}_h^k \underline{\boldsymbol{v}}_h \, q_h$.

and where s_h is a classic HHO stabilization function satisfying a power-framed assumption similar to that of $\boldsymbol{\sigma}$.

Assumptions on s_h

Assumption. *(Stabilization function s_h)*

- For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, $\mathrm{s}_h(\underline{\boldsymbol{v}}_h, \cdot)$ is linear.
- Polynomial consistency. For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$ and $\boldsymbol{w} \in \mathbb{P}^{k+1}(T, \mathbb{R}^d)$,

$$\mathrm{s}_T(\underline{\mathbf{I}}_T^k \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0.$$

- Stability and boundedness. For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\|\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h\|_{L^r(\Omega,\mathbb{R}^{d\times d})}^r + \mathrm{s}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) \simeq \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}^r.$$

- r-power-framed. For all $\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{U}}_h^k$ with $\underline{\boldsymbol{e}}_h \coloneqq \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{w}}_h$,

$$|\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) - \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)| \lesssim (\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{w}}_h))^{\frac{r-\tilde{r}}{r}} \mathrm{s}_h(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{e}}_h)^{\frac{\tilde{r}-1}{r}} \mathrm{s}_h(\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h)^{\frac{1}{r}},$$

$$\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{e}}_h) - \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{e}}_h) \gtrsim (\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{w}}_h))^{\frac{\tilde{r}-2}{r}} \mathrm{s}_h(\underline{\boldsymbol{e}}_h, \underline{\boldsymbol{e}}_h)^{\frac{r+2-\tilde{r}}{r}}.$$

Properties of a_h

Lemma. *(Properties of a_h)*

- Hölder continuity. For all $\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{U}}_h^k$, setting $\underline{\boldsymbol{e}}_h \coloneqq \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{w}}_h$,

$$|\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) - \mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)| \lesssim \sigma_{\mathrm{hc}} \left(\sigma_{\mathrm{de}}^r + \|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h}^r + \|\underline{\boldsymbol{w}}_h\|_{\varepsilon,r,h}^r\right)^{\frac{r-\tilde{r}}{r}} \|\underline{\boldsymbol{e}}_h\|_{\varepsilon,r,h}^{\tilde{r}-1} \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}.$$

- Strong monotonicity. For all $\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{U}}_h^k$, setting $\underline{\boldsymbol{e}}_h \coloneqq \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{w}}_h$,

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{e}}_h) - \mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{e}}_h) \gtrsim \sigma_{\mathrm{sm}} \left(\sigma_{\mathrm{de}}^r + \|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h}^r + \|\underline{\boldsymbol{w}}_h\|_{\varepsilon,r,h}^r\right)^{\frac{2-\tilde{r}}{r}} \|\underline{\boldsymbol{e}}_h\|_{\varepsilon,r,h}^{r+2-\tilde{r}}.$$

Properties of b_h

Lemma. *(Properties of b_h)*

- Inf-sup stability. For all $q_h \in P_h^k$,

$$\|q_h\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h).$$

- Fortin operator. For all $\boldsymbol{v} \in W^{1,r}(\Omega,\mathbb{R}^d)$,

$$\|\underline{\mathbf{I}}_h^k \boldsymbol{v}\|_{\varepsilon,r,h} \lesssim |\boldsymbol{v}|_{W^{1,r}(\Omega,\mathbb{R}^d)},$$

$$\mathrm{b}_h(\underline{\mathbf{I}}_h^k \boldsymbol{v}, q_h) = b(\boldsymbol{v}, q_h) \qquad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h,\mathbb{R}).$$

Well-posedness and a priori bounds

Theorem. *(Well-posedness and a priori bounds)*

There exists a unique solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ to the discrete weak formulation. Additionally, the following a priori bounds hold:

$$\|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h} \lesssim \left(\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r-1}} + \left(\sigma_{\mathrm{de}}^{2-\tilde{r}}\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r+1-\tilde{r}}},$$

$$\|p_h\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sigma_{\mathrm{hc}}\left(\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)} + \sigma_{\mathrm{de}}^{|r-2|(\tilde{r}-1)}\left(\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{\tilde{r}-1}{r+1-\tilde{r}}}\right).$$

Error estimate

Theorem. *(Error estimate)*

Let $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ solve the continuous and discrete weak formulations, respectively. Assume also

- $\boldsymbol{u} \in W^{k+2,r}(\mathcal{T}_h, \mathbb{R}^d)$,
- $\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) \in W^{1,r'}(\Omega, \mathbb{R}_{\mathrm{s}}^{d\times d}) \cap W^{(k+1)(\tilde{r}-1),r'}(\mathcal{T}_h, \mathbb{R}_{\mathrm{s}}^{d\times d})$,
- $p \in W^{1,r'}(\Omega, \mathbb{R}) \cap W^{(k+1)(\tilde{r}-1),r'}(\mathcal{T}_h, \mathbb{R})$.

Then,

$$\|\underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u}\|_{\varepsilon,r,h} \lesssim C_1 h^{\frac{(k+1)(\tilde{r}-1)}{r+1-\tilde{r}}},$$

$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim C_2 h^{\frac{(k+1)(\tilde{r}-1)^2}{r+1-\tilde{r}}},$$

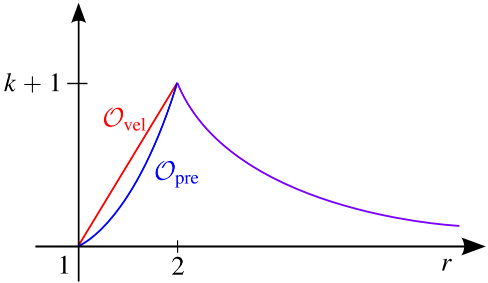
where $C_1, C_2 \in [0, +\infty)$ depend only on $\boldsymbol{u}$,p,$\boldsymbol{f}$,σ_{hc},σ_{sm}, and σ_{de}.

Asymptotic convergence rates

Asymptotic convergence rates:

$$\mathcal{O}_{\mathrm{vel}} \coloneqq \frac{(k+1)(\tilde{r}-1)}{r+1-\tilde{r}} = \begin{cases} (k+1)(r-1) & \text{if } r < 2 \\ \dfrac{k+1}{r-1} & \text{if } r \geqslant 2 \end{cases},$$

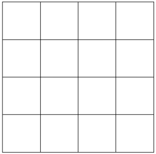
$$\mathcal{O}_{\mathrm{pre}} \coloneqq \frac{(k+1)(\tilde{r}-1)^2}{r+1-\tilde{r}} = \begin{cases} (k+1)(r-1)^2 & \text{if } r < 2 \\ \dfrac{k+1}{r-1} & \text{if } r \geqslant 2 \end{cases}.$$



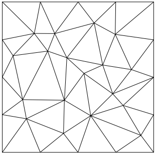
Numerical results 2D

SpaFEDTe library (2D and 3D)

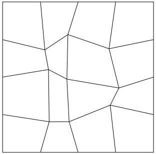
- We consider $\Omega = (0,1)^2$ and the following three mesh families.



Cartesian



distorted triangular



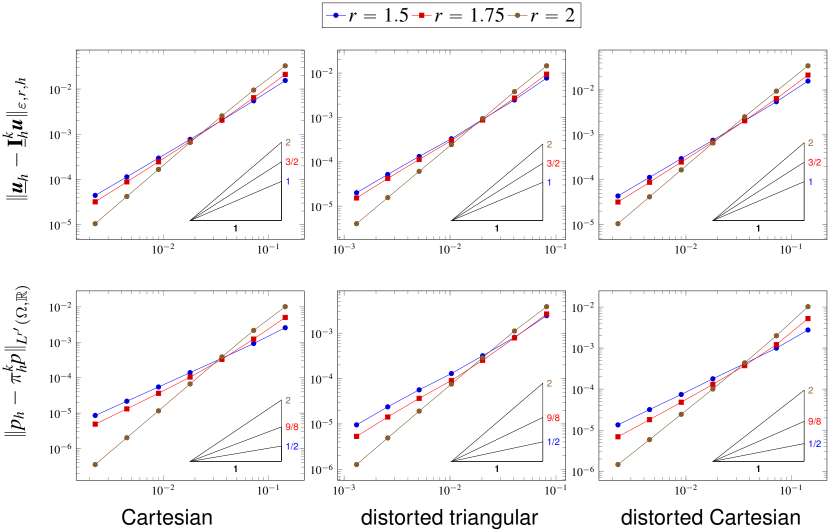
distorted Cartesian

- For a well chosen $\boldsymbol{f}$, the exact velocity $\boldsymbol{u}$ and pressure p are given such that for all $(x, y) \in \Omega$,

$$\boldsymbol{u}(x,y) = \left(\sin\left(\tfrac{\pi}{2}x\right)\cos\left(\tfrac{\pi}{2}y\right), -\cos\left(\tfrac{\pi}{2}x\right)\sin\left(\tfrac{\pi}{2}y\right)\right),$$
$$p(x,y) = \sin\left(\tfrac{\pi}{2}x\right)\sin\left(\tfrac{\pi}{2}y\right) - \tfrac{4}{\pi^2}.$$

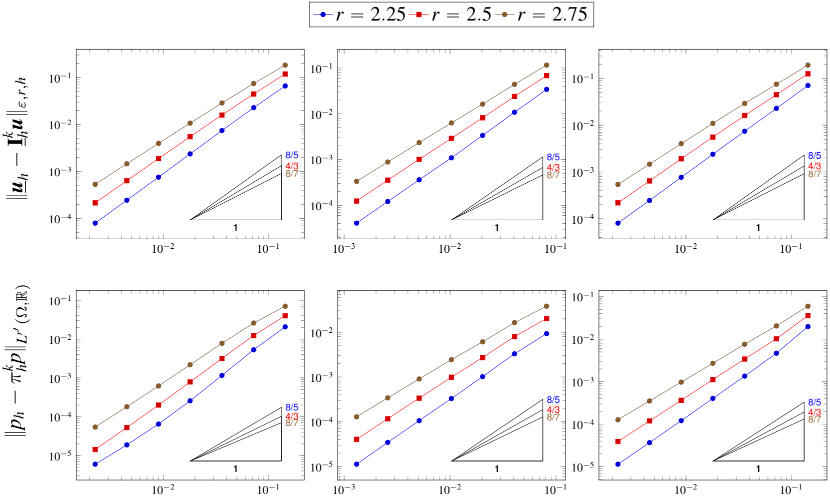
Numerical results 2D

Results for $k = 1$ and $r \leqslant 2$, so $\mathcal{O}_{\mathrm{vel}} = 2(r-1)$ and $\mathcal{O}_{\mathrm{pre}} = 2(r-1)^2$.



Numerical results 2D

Results for $k = 1$ and $r \geqslant 2$, so $\mathcal{O}_{\text{vel}} = \mathcal{O}_{\text{pre}} = \frac{2}{r-1}$.



Cartesian distorted triangular distorted Cartesian

Next steps

Next steps:

- In-depth analysis of the asymptotic convergence rates.
- Look for convergence by compactness.
- Extend the analysis to the Navier–Stokes equations.
- Moving to rheopecty and thixotropic fluids: r evolves over time.

Thank you very much for your attention!

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