Improved error estimates for Hybrid High-Order discretizations of Leray-Lions problems

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Outline

1. The Hybrid High-Order method
2. The Leray–Lions problem
3. Discretization with the HHO method
4. Numerical results

Hybrid High-Order

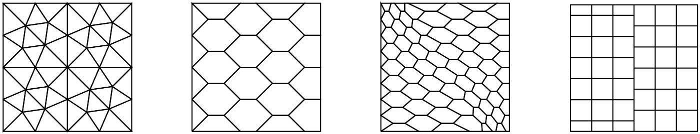
Hybrid: two kinds of unknowns located on the mesh and its skeleton.

High-Order: the unknowns live in broken polynomial spaces of degree $k \in \mathbb{N}$.

General features

This approach possesses several attractive features:

- Arbitrary approximation order ($k \geqslant 0$).
- Formulation valid for arbitrary space dimension.
- Seamless treatment of nonconforming mesh refinement.



- Moderate computational costs thanks to static condensation.
- Inf-sup stable discretizations.

References

Book.

Daniele A. Di Pietro and Jérôme Droniou.
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Subject matter

Article.

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Outline

The Leray–Lions problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, denote a bounded, connected polytopal open set with Lipschitz boundary $\partial\Omega$.

The Leray–Lions problem reads: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\boldsymbol{\nabla}\cdot \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla} u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where,

- $u : \Omega \to \mathbb{R}$ is the potential,
- $f : \Omega \to \mathbb{R}$ is a volumetric force term,
- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is the flux function.

Applications: glaciology, airfoil design, turbulent porous media

Assumptions on the flux function $\boldsymbol{\sigma}$

Assumption. *(Flux function)*

There exists $p \in (1, 2]$ such that:

- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable.
- $\boldsymbol{\sigma}(\cdot, \mathbf{0}) \in L^{p'}(\Omega)^d$ a.e. in Ω where $p' \coloneqq \frac{p}{p-1}$.
- There is $\delta \in L^p(\Omega, [0, +\infty))$ and $\sigma_{\text{hc}}, \sigma_{\text{sm}} \in (0, +\infty)$ such that

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})| \leqslant \sigma_{\text{hc}} \left(\delta(\boldsymbol{x})^p + |\boldsymbol{\tau}|^p + |\boldsymbol{\eta}|^p\right)^{\frac{p-2}{p}} |\boldsymbol{\tau} - \boldsymbol{\eta}|,$$

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \sigma_{\text{sm}} \left(\delta(\boldsymbol{x})^p + |\boldsymbol{\tau}|^p + |\boldsymbol{\eta}|^p\right)^{\frac{p-2}{p}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^2,$$

  for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Omega$.

Examples of flux function

The following flux functions satisfy the previous assumption:

- The p-Laplace function s.t. for all $\boldsymbol{\tau} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Omega$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = |\boldsymbol{\tau}|^{p-2}\boldsymbol{\tau}.$$

- The Carreau-Yasuda function s.t. for all $\boldsymbol{\tau} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Omega$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = \mu(\boldsymbol{x}) \left(\delta(\boldsymbol{x})^{a(\boldsymbol{x})} + |\boldsymbol{\tau}|^{a(\boldsymbol{x})} \right)^{\frac{p-2}{a(\boldsymbol{x})}} \boldsymbol{\tau},$$

  where $\mu : \Omega \to [\mu_-, \mu_+]$ and $a : \Omega \to [a_-, a_+]$ are measurable functions with $\mu_-, \mu_+, a_-, a_+ \in (0, +\infty)$.

Weak formulation

Assuming $f \in L^{p'}(\Omega)$, the weak formulation of the Leray–Lions problem reads: Find $u \in W_0^{1,p}(\Omega)$ such that

$$a(u,v) = \int_\Omega f\, v \qquad \forall v \in W_0^{1,p}(\Omega),$$

where for all $v, w \in W_0^{1,p}(\Omega)$,

$$a(w,v) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla} w) \cdot \boldsymbol{\nabla} v.$$

Proposition. *(Well-posedness and a priori bound)*

There exists a unique solution $u \in W_0^{1,p}(\Omega)$ to the weak formulation. Additionally, the following a priori bound hold:

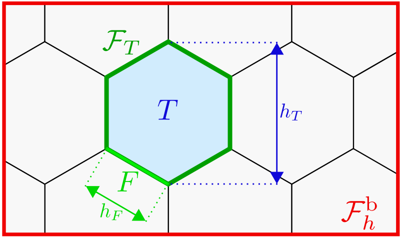
$$\|\boldsymbol{\nabla} u\|_{L^p(\Omega)^d} \leqslant C\Big(\sigma_{\mathrm{sm}}, \|f\|_{L^{p'}(\Omega)}, \|\delta\|_{L^p(\mathcal{T}_h)}, \Omega, p\Big).$$

Outline

Mesh and notations

A polytopal mesh of Ω is a couple $\mathcal{M}_h \coloneqq (\mathcal{T}_h, \mathcal{F}_h)$ where

- $\mathcal{T}_h$ contains polytopal elements T with diameter h_T,
- $\mathcal{F}_h$ contains hyperplanar faces F with diameter h_F,
- $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$, $\max_{T\in\mathcal{T}_h} h_T = h$, plus some geometrical requirements



Subsets of $\mathcal{F}_h$:

- $\mathcal{F}_h^{\mathrm{b}} \coloneqq \{F \in \mathcal{F}_h \ :\ F \subset \partial\Omega\}$,
- $\mathcal{F}_T \coloneqq \{F \in \mathcal{F}_h \ :\ F \subset \partial T\}$ for all $T \in \mathcal{T}_h$.

Discrete spaces and notations

Let $k \geqslant 0$ be the polynomial degree of the HHO method.

- For all $T \in \mathcal{T}_h$, we define the discrete local space:

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right).$$

  We use the discrete notation $\underline{v}_T := (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$.

- We define the discrete global space:

$$\underline{U}_h^k := \prod_{T \in \mathcal{T}_h} \underline{U}_T^k.$$

  We use the discrete notation $\underline{v}_h := (\underline{v}_T)_{T \in \mathcal{T}_h} \in \underline{U}_h^k$.

- We define the interpolation operator $\underline{\mathrm{I}}_h^k : W^{1,1}(\Omega) \to \underline{U}_h^k$ s.t.

$$\underline{\mathrm{I}}_h^k v := (\pi_T^k v_{|_T}, (\pi_F^k v_{|_F})_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h}.$$

Spaces and norms of discrete unknowns

The discrete space containing potential unknowns is defined by

$$\underline{U}_{h,0}^k \coloneqq \left\{\underline{v}_h \in \underline{U}_h^k \;:\; v_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}.$$

We endow $\underline{U}_{h,0}^k$ with the norm $\|\cdot\|_{1,p,h}$ defined s.t. for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|\underline{v}_h\|_{1,p,h}^p \coloneqq \sum_{T\in\mathcal{T}_h}\left(\|\boldsymbol{\nabla} v_T\|_{L^p(T)^d}^p + \sum_{F\in\mathcal{F}_T} h_F^{1-p}\|v_F - v_T\|_{L^p(F)}^p\right).$$

Discrete operators

For all $T \in \mathcal{T}_h$, we define:

- The local gradient reconstruction $\mathbf{G}_T^k : \underline{U}_T^k \longrightarrow \mathbb{P}^k(T)^d$ s.t. for all $\underline{v}_T \in \underline{U}_T^k$,

$$\int_T \mathbf{G}_T^k \underline{v}_T \cdot \boldsymbol{\tau} = \int_T \boldsymbol{\nabla} v_T \cdot \boldsymbol{\tau} + \sum_{F \in \mathcal{F}_T} \int_F (v_F - v_T)(\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T)^d.$$

- The local potential reconstruction $\mathrm{r}_T^{k+1} : \underline{U}_T^k \longrightarrow \mathbb{P}^{k+1}(T)$ s.t.

$$\int_T (\boldsymbol{\nabla} \mathrm{r}_T^{k+1} \underline{v}_T - \mathbf{G}_T^k \underline{v}_T) \cdot \boldsymbol{\nabla} w = 0 \qquad \forall w \in \mathbb{P}^{k+1}(T),$$

$$\int_T \mathrm{r}_T^{k+1} \underline{v}_T = \int_T v_T.$$

- The boundary residual operator $\Delta_{\partial T}^k : \underline{U}_T^k \to L^p(\partial T)$ s.t. for all $\underline{v}_T \in \underline{U}_T^k$,

$$(\Delta_{\partial T}^k \underline{v}_T)_{|F} \coloneqq \frac{1}{h_T} \left[\pi_F^k (\mathrm{r}_T^{k+1} \underline{v}_T - v_F) - \pi_T^k (\mathrm{r}_T^{k+1} \underline{v}_T - v_T) \right] \qquad \forall F \in \mathcal{F}_T.$$

Discrete weak formulation

The discrete weak formulation reads: Find $(\underline{u}_h, p_h) \in \underline{U}_{h,0}^k \times P_h^k$ s.t.

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f\ v_h \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k,$$

where, for all $\underline{w}_h, \underline{v}_h \in \underline{U}_h^k$,

$$\mathrm{a}_h(\underline{w}_h, \underline{v}_h) \coloneqq \sum_{T \in \mathcal{T}_h} \left(\int_T \boldsymbol{\sigma}(\cdot, \mathbf{G}_T^k \underline{w}_T) \cdot \mathbf{G}_T^k \underline{v}_T + h_T \int_{\partial T} \mathrm{S}_T(\cdot, \Delta_{\partial T}^k \underline{w}_T)\ \Delta_{\partial T}^k \underline{v}_T \right),$$

and where for all $T \in \mathcal{T}_h$, S_T is a local stabilization function satisfying an assumption similar to that of $\boldsymbol{\sigma}$.

Assumptions on s_T

Assumption. *(Stabilization functions)*

There exists $\zeta \in L^p(\partial\mathcal{M}_h; [0, +\infty))$, where $\partial\mathcal{M}_h \coloneqq \bigcup_{F \in \mathcal{F}_h} \overline{F}$, such that for all $T \in \mathcal{T}_h$:

- $\mathrm{S}_T : \partial T \times \mathbb{R} \to \mathbb{R}$ is measurable.
- $\mathrm{S}_T(\boldsymbol{x}, 0) = 0$ for a.e. $\boldsymbol{x} \in \Omega$.
- For all $v, w \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \partial T$,

$$|\mathrm{S}_T(\boldsymbol{x}, w) - \mathrm{S}_T(\boldsymbol{x}, v)| \lesssim \sigma_{\mathrm{hc}} \left(\zeta(\boldsymbol{x})^p + |w|^p + |v|^p\right)^{\frac{p-2}{p}} |w - v|,$$
$$\left(\mathrm{S}_T(\boldsymbol{x}, w) - \mathrm{S}_T(\boldsymbol{x}, v)\right)(w - v) \gtrsim \sigma_{\mathrm{sm}} \left(\zeta(\boldsymbol{x})^p + |w|^p + |v|^p\right)^{\frac{p-2}{p}} |w - v|^2.$$

Stabilization functions that match the above assumption can be obtained setting for all $T \in \mathcal{T}_h$,

$$\mathrm{S}_T(\boldsymbol{x}, w) = \left(\zeta(\boldsymbol{x})^p + |w|^p\right)^{\frac{p-2}{p}} w \qquad \forall w \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega.$$

Well-posedness and a priori bound

Lemma. *(Strong monotonicity of a_h)*

For all $\underline{v}_h, \underline{w}_h \in \underline{U}_h^k$, setting $\underline{e}_h \coloneqq \underline{v}_h - \underline{w}_h$,

$$\|\underline{e}_h\|_{1,p,h}^2 \lesssim \sigma_{\mathrm{sm}}^{-1} \left(\|\delta\|_{L^p(\Omega)}^p + \|\zeta\|_{L^p(\partial\mathcal{M}_h)}^p + \|\underline{v}_h\|_{1,p,h}^p + \|\underline{w}_h\|_{1,p,h}^p \right)^{\frac{2-p}{p}} \times \left(\mathrm{a}_h(\underline{v}_h, \underline{e}_h) - \mathrm{a}_h(\underline{w}_h, \underline{e}_h) \right)$$

Theorem. *(Well-posedness and a priori bound)*

There exists a unique solution $\underline{u}_h \in \underline{U}_{h,0}^k$ to the discrete weak formulation. Additionally, the following a priori bound hold:

$$\|\underline{u}_h\|_{1,p,h} \leqslant C\Big(\sigma_{\mathrm{sm}}, \|f\|_{L^{p'}(\Omega)}, \|\delta\|_{L^p(\mathcal{T}_h)}, \|\zeta\|_{L^p(\partial\mathcal{M}_h)}, \Omega, p\Big).$$

Error estimate

Theorem. *(Error estimate)*

Let $u \in W_0^{1,p}(\Omega)$ and $\underline{u}_h \in \underline{U}_{h,0}^k$ solve the continuous and discrete weak formulations, respectively. Assume

- $u \in W^{k+2,p}(\mathcal{T}_h)$,
- $\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla} u) \in W^{1,p'}(\Omega)^d \cap W^{k+1,p'}(\mathcal{T}_h)^d$.

Then, setting $\underline{e}_h := \underline{u}_h - \underline{\mathrm{I}}_h^k u$, we have

$$\|\underline{e}_h\|_{1,p,h} \leqslant h^{k+1} C_1 + \left[\sum_{T \in \mathcal{T}_h} \left[\min(\eta_T; 1)^{2-p} h_T^{(k+1)(p-1)} |u|_{W^{k+2,p}(T)}^{p-1} \right]^{p'} \right]^{\frac{1}{p'}} C_2,$$

where $C_1, C_2 \in [0, +\infty)$ depends only on $u, f, \boldsymbol{\sigma}$, and where

$$\eta_T := h_T^{k+1} \left(|T|^{-\frac{1}{p}} |u|_{W^{k+2,p}(T)} \right) \mathfrak{D}_T^{-1}$$

with $\mathfrak{D}_T := \min \left(\operatorname*{ess\,inf}_T (\delta + |\boldsymbol{\nabla} u|) ; \operatorname*{ess\,inf}_{\partial T} \zeta \right).$

Local convergence rates

For any $T \in \mathcal{T}_h$, we define the local convergence rate $\mathcal{O}_T^k$ and determines it among 3 cases according to η_T:

- If $\eta_T \geqslant 1$ (degenerate case), then
$$\mathcal{O}_T^k = (k+1)(p-1).$$
- If $\eta_T \leqslant h_T^{k+1}\left(|T|^{-\frac{1}{p}}|u|_{W^{k+2,p}(T)}\right)$ (non-degenerate case), then
$$\mathcal{O}_T^k = k+1.$$
- Otherwise,
$$\mathcal{O}_T^k \in ((k+1)(p-1), k+1).$$

The non-degenerate case is equivalent to $\mathfrak{D}_T \geqslant 1$. Moreover, we can choose $\zeta \geqslant 1$, so the non-degenerate case corresponds to

$$\operatorname*{ess\,inf}_T\, (\delta + |\nabla u|) \geqslant 1.$$

Global convergence rates

The global convergence rate is naturally $\mathcal{O}_h^k \coloneqq \min_{T\in\mathcal{T}_h} \mathcal{O}_T^k$. We can have an analogous disjunction of cases by defining $\eta_h = \max_{T\in\mathcal{T}_h} \eta_T$:

- If $\eta_h \geqslant 1$ (degenerate case), then
$$\mathcal{O}_h^k = (k+1)(p-1).$$
- If $\eta_h \leqslant h^{k+1}|u|_{W^{k+2,p}(\mathcal{T}_h)}$ (non-degenerate case), then
$$\mathcal{O}_h^k = k+1.$$
- Otherwise,
$$\mathcal{O}_h^k \in ((k+1)(p-1), k+1).$$

As at the local level, the non-degenerate case is equivalent to

$$\operatorname*{ess\,inf}_{\Omega} (\delta + |\boldsymbol{\nabla} u|) \geqslant 1.$$

Outline

Numerical results

- We consider $\Omega = (0,1)^2$ and a triangular mesh family.
- We choose the following Carreau-Yasuda flux function:

$$\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) = (\delta(\boldsymbol{x}) + |\boldsymbol{\tau}|)^{p-2}\boldsymbol{\tau} \qquad \forall \boldsymbol{\tau} \in \mathbb{R}^2 \quad \text{a.e. } \boldsymbol{x} \in \Omega,$$

  If $\delta = 0$ a.e. in Ω, then $\boldsymbol{\sigma}$ becomes the p-Laplace function.

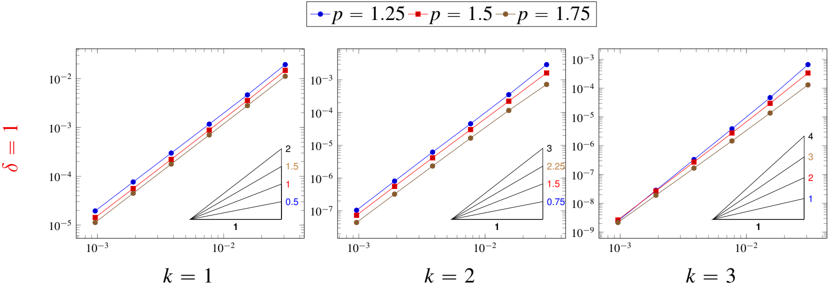
- The force term f is given by u and $\boldsymbol{\sigma}$.

We consider $\delta \in \{1, 10^{-2}\}$ and define the potential such that

$$u(x,y) = \sin(\pi x)\sin(\pi y) \quad \forall (x,y) \in \Omega.$$

We have $\eta_h \simeq h^{k+1}(2^{\frac{k-1}{2}}\pi^k)\delta^{-1}$.

The convergence rate should switches from $(k+1)(p-1)$ to $(k+1)$ as h is small enough compared to δ.



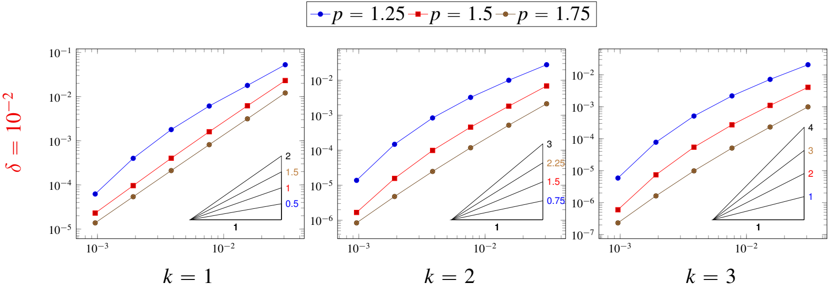
Non-degenerate flux

We consider $\delta \in \{1, 10^{-2}\}$ and define the potential such that

$$u(x, y) = \sin(\pi x) \sin(\pi y) \quad \forall (x, y) \in \Omega.$$

We have $\eta_h \simeq h^{k+1}(2^{\frac{k-1}{2}}\pi^k)\delta^{-1}$.

The convergence rate should switches from $(k+1)(p-1)$ to $(k+1)$ as h is small enough compared to δ.

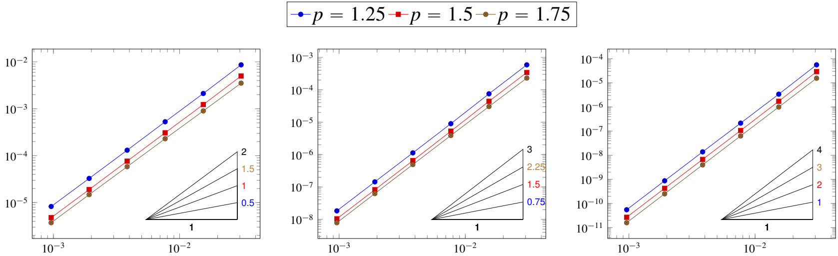


Non-degenerate potential

We set $\delta = 0$ and we define the potential such that

$$u(x, y) = \sin(\pi x)\sin(\pi y) + (\pi + 1)(x + y) \quad \forall (x, y) \in \Omega.$$

We have $\operatorname{ess\,inf}_\Omega (\delta + |\nabla u|) = 1$, so we should observe a constant convergence rate of $(k + 1)$.



$k = 1$ $k = 2$ $k = 3$

Non-degenerate flux-potential couple

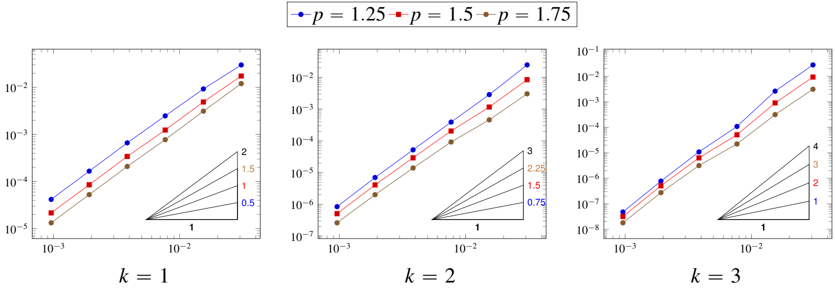
The potential is defined such that

$$u(x,y) = \sin(\pi x)\sin(\pi y) \quad \forall (x,y) \in \Omega.$$

Let $(\boldsymbol{x}_i)_{1\leqslant i\leqslant 5}$ be the points where $\boldsymbol{\nabla} u$ vanishes, we set

$$\delta(\boldsymbol{x}) = \sum_{i=1}^{5} \mathbf{1}_{\{|\boldsymbol{x}-\boldsymbol{x}_i|<\frac{1}{5}\}} \exp\left(1 - \frac{1}{1-25|\boldsymbol{x}-\boldsymbol{x}_i|^2}\right) \quad \forall \boldsymbol{x} \in \Omega.$$

We infer $\operatorname{ess\,inf}_\Omega (\delta + |\boldsymbol{\nabla} u|) = 1$ which infer a $(k+1)$ convergence rate.

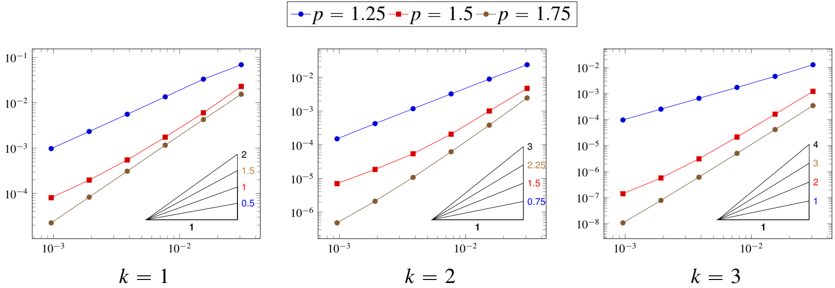


Degenerate problem

We set $\delta = 0$ and we choose the following potential:

$$u(x,y) = \frac{1}{10}\exp\left(-10\left(|x-0.5|^{p+\frac{k+2}{4}} + |y-0.5|^{p+\frac{k+2}{4}}\right)\right) \quad \forall (x,y) \in \Omega.$$

We have $\operatorname{ess\,inf}_\Omega(\delta + |\nabla u|) = 0$, so $\eta_h = +\infty$ and the order of convergence expected is $(k+1)(p-1)$.



Conclusion:

- We have presented and analysed a Hybrid High-Order scheme of arbitrary order k, for a non-linear model that generalises the p-Laplace equation with $p \in (1, 2]$.
- For a degenerate model we recover the known rates of convergence in $(k + 1)(p - 1)$, except when p is small where the convergence appears to be faster than expected. Very recently, [2] proved convergence in $\frac{k+1}{3-p}$.
- An optimal rate of $(k + 1)$, identical to the rate for linear models, is obtained when the model is not degenerate.
- These regimes are driven by a dimensionless number, and intermediate regimes are also identified.

Thank you for your attention!

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